Electrodynamics Problems

This is a collection of problems in electrodynamics for undergraduate physics students, put together by Prof. Dr. Ana-Sunčana Smith and her group.
Some of these exercises have been conceived by:

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A majority of the exercises have been taken from:

- J. D. Jackson, *Classical Electrodynamics*
- D. J. Griffiths, *Introduction to Electrodynamics*
- W. Greiner, *Classical Electrodynamics*
- Y. K. Lim (ed.), *Problems and Solutions on Electromagnetism*
- V. V. Batygin and I. N. Toptygin, *Problems in Electrodynamics*

All users of this collection are requested to kindly report any errors and omissions to the Smith group.

Mathematics

Problem 1.1 *Nabla-operator*

a) Consider the function $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}, \vec{r} \longmapsto \phi(\vec{r})$. Here $\vec{r} = (x, y, z)^T$ represents the position vector. Calculate the vector (the "gradient" of ϕ)

$$\vec{G}(\vec{r}) = \frac{\partial \phi}{\partial x}\vec{e}_x + \frac{\partial \phi}{\partial y}\vec{e}_y + \frac{\partial \phi}{\partial z}\vec{e}_z$$

for the function $\phi(\vec{r}) = r^2$ and draw the vector field $\vec{G}(\vec{r})$ on the plane $z = 0$.

b) Consider the function $\vec{A} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \vec{r} \longmapsto \vec{A}(\vec{r}) = A_x(\vec{r})\vec{e}_x + A_y(\vec{r})\vec{e}_y + A_z(\vec{r})\vec{e}_z$. Calculate the scalar function (the "divergence" of $\vec{A}$)

$$D(\vec{r}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

for the function $\vec{A}(\vec{r}) = (\vec{c} \cdot \vec{r})\vec{r}$, where $\vec{c}$ is a constant vector (it is independent of $\vec{r}$).

c) Calculate the vector ("curl" of $\vec{A}$)

$$\vec{R}(\vec{r}) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e}_z$$

for the function $\vec{A}(\vec{r})$ given in the part b).

d) Calculate the functional determinant $\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)}\right|$ where (ρ, φ, z) are the usual cylindrical coordinates.

Problem 1.2 *Field lines*

Field lines are integral curves tangent to the vector field, i.e., up to reparametrization they fulfill

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{x}(\tau) = \vec{E}(\vec{x}(\tau)) \,, \qquad \vec{x}(\tau = 0) = \vec{x}_0 \,.$$

Consider the vector field $\vec{E}(\vec{x}) = (y, x, 0)$. Verify that $\vec{E}$ is irrotational, i.e., $\vec{\nabla} \times \vec{E} = 0$, and construct a scalar potential $\varphi(\vec{x})$ by evaluating the line integral

$$\varphi(\vec{x}) = -\int_{\mathcal{C}} \vec{E} \cdot \mathrm{d}\vec{l}\,,$$

for curves $\mathcal{C}$ connecting the origin with the point $\vec{x} = (x, y, z)$. Discuss the equipotential surfaces and calculate the field lines corresponding to $\vec{E}(\vec{x})$.

Problem 1.3 *Differential operators in cylindrical coordinates*

In cartesian coordinates the nabla operator is defined as follows:

$$\vec{\nabla} = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}$$

Cylindrical coordinates (ρ, ϕ, z) are defined as follows:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

a) Transform the nabla operator into cylindrical coordinates. Proceed with the following steps:

- Use the chain rule to write the partial derivatives $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ in terms of $\partial/\partial\rho$, $\partial/\partial\phi$, $\partial/\partial z$
- Write $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ in terms of $\vec{e}_\rho$, $\vec{e}_\phi$, $\vec{e}_z$

Result: $\vec{\nabla} = \vec{e}_\rho \frac{\partial}{\partial\rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}$

b) Calculate for a vector field $\vec{A}$ the expression $\vec{\nabla} \cdot \vec{A}$

Result: $\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$

Problem 1.4 *General relations and the curl*

a) Prove the relation

$$\vec{\nabla} \cdot (\vec{V} \times \vec{W}) = \vec{W} \cdot \vec{\nabla} \times \vec{V} - \vec{V} \cdot \vec{\nabla} \times \vec{W}$$

for two vector fields $\vec{V}$ and $\vec{W}$ through the analysis in the Cartesian coordinates.

b) Prove the following relation for a scalar field ϕ:

$$\vec{\nabla} \cdot (\phi\vec{V}) = \phi(\vec{\nabla} \cdot \vec{V}) + \vec{V} \cdot (\nabla\phi).$$

c) Derive the expression for the curl in the usual cylindrical coordinates.

d) Find the vector field $\vec{V} = V_\rho(\rho, \varphi, z)\vec{e}_\rho$, which satisfies the equation $\vec{\nabla} \times \vec{V} = \varphi\vec{e}_z$.

Problem 1.5 *Differential operators in cylindrical coordinates*

In cartesian coordinates the nabla operator is defined as follows:

$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

Cylindrical coordinates (ρ, ϕ, z) are defined as follows:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

Let $f(\rho, \phi, z)$ be a scalar field. Calculate $\Delta f := \vec{\nabla} \cdot (\vec{\nabla} f)$ in cylindrical coordinates.
Result: $\Delta f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

Problem 1.6 *Stokes' and Gauß' theorems*

a) Verify Stokes' theorem for the vector field

$$\vec{V} = (4x/3 - 2y)\vec{e}_x + (3y - x)\vec{e}_y$$

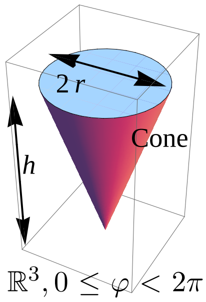
and the surface $A = \left\{\vec{r}\,|(x/3)^2 + (y/2)^2 \leq 1 \quad \text{and} \quad z = 0\right\}$.

b) Verify Gauß' theorem for the vector field

$$\vec{V} = ax\vec{e}_x + by\vec{e}_y + cz\vec{e}_z$$

and the sphere $x^2 + y^2 + z^2 \leq R^2$.

Problem 1.7 *Stokes' theorem*



Consider the axially symmetric vector field $\vec{B}(\vec{r}) = F_M\,(x, y, -2z)^T$.

a) Calculate and sketch the field lines in the x-z plane.

b) Calculate the surface integral

$$\Phi = \int_\Sigma \vec{B} \cdot \mathrm{d}\vec{F}$$

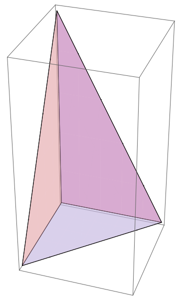
for

(i) a circular area $\Sigma_1 = \{(x, y, z)^T \in \mathbb{R}^3; x^2 + y^2 \le a^2, z = h\}$

(ii) and for the surface of a cone $\Sigma_2 = \{((za/h)\cos\varphi, (za/h)\sin\varphi, z)^T \in \mathbb{R}^3, 0 \le \varphi < 2\pi, 0 \le z \le h\}$.

c) Show that the vector field is solenoidal, div $\vec{B}(\vec{r}) = 0$. Find a suitable vector potential $\vec{A}(\vec{r})$, $\vec{B} = \vec{\nabla} \times \vec{A}$. Calculate the line integral $\int_{\partial\Sigma} \vec{A} \cdot \mathrm{d}\vec{r}$ and verify Stokes' theorem.

Problem 1.8 *Gauß' theorem*



Consider the pyramid $\Omega = \{(x, y, z)^T \in \mathbb{R}^3 : 0 \le z \le 2 - 2x - 2y,\ 0 \le y \le 1 - x,\ 0 \le x \le 1\}$. and the vector field $\vec{B}(\vec{r}) = G_M(y, x, 0)$.

a) Calculate the volume integral $\int_\Omega \mathrm{div}\ \vec{B}\mathrm{d}V$.

b) Evaluate the surface integral $\int_{\partial\Omega} \vec{B} \cdot \mathrm{d}\vec{F}$ and confirm Gauss' theorem.

Problem 1.9 *General integral theorems*

The particular importance of the integral theorems of Gauß and Stokes arises from the transition from a volume integral to a surface integral and from a surface integral to a curvilinear integral, respectively. These integral theorems are special forms of more general theorems. Show that more general integral theorems can be derived from the theorems of Gauß and Stokes by an appropriate choice of $\vec{A}$ in the following generalized equation:

$$\int_V \nabla \circ \hat{A}\, \mathrm{d}V = \int_a \vec{n} \circ \hat{a}\, \mathrm{d}a$$

Prove especially

$$\int_V \mathrm{d}V \nabla \circ = \int_{a(V)} \mathrm{d}\vec{a} \circ$$

and

$$\int_a (\mathrm{d}\vec{a} \times \nabla) \circ = \oint_{C_a} \mathrm{d}\vec{r} \circ$$

Remark

Here, the operation $\circ$ may be a scalar product $(\cdot)$ with a vector (then $\hat{A}$ is a vector so that the whole makes sense), the gradient of a scalar (then $\hat{A}$ is a scalar), or the cross product $(\times)$ with a vector (then $\hat{A}$ is a vector).

Problem 1.10 *Debye potentials*

A vector field $\vec{B}_t$ that can be represented as $\vec{B}_t = \vec{\nabla} \times (\vec{r}\psi)$ with the scalar potential $\psi = \psi(\vec{r})$ is referred to as toroidal. Similarly, a vector field $\vec{B}_p(\vec{r})$ that can be represented as $\vec{B}_p = \vec{\nabla} \times \left[\vec{\nabla} \times (\vec{r}\psi)\right]$ with the scalar potential $\chi = \chi(\vec{r})$ is called poloidal. The scalar fields $\psi(\vec{r}), \chi(\vec{r})$ are known as Debye potentials. A theorem from vector analysis states that a solenoidal vector field $\vec{B}(\vec{r})$, div $\vec{B} = 0$ can be represented as a superposition of a toroidal and a poloidal vector field $\vec{B} = \vec{B}_t + \vec{B}_p$.z

a) Show that toroidal and poloidal vector fields are solenoidal. Conclude that solenoidal vector fields can also be represented in terms of suitable vector potental $\vec{A}(\vec{r})$ as $\vec{B} = \vec{\nabla} \times \vec{A}$.

b) Demonstrate that a toroidal field has no radial component and consequently its field lines are confined to spherical surfaces. What do the field lines of toroidal field look like in the case of an axial symmetry? Show that a poloidal field is perpendicular to a toroidal field. Show that for axially symmetric poloidal fields the field lines are in the meridian planes.

c) Sketch the field lines corresponding to the poloidal potential $\chi(\vec{r}) = \vec{m} \cdot \vec{r}/r^2$.

d) Show that the curl of a toroidal field is poloidal and the curl of a poloidal field is toroidal.

Problem 1.11 *Levi-Civita-tensor*

The Levi-Civita-tensor is defined as follows:

$$\varepsilon_{\alpha\beta\gamma} = \begin{cases} 1 & \text{when } (\alpha, \beta, \gamma) \text{ is an even permutation of (1,2,3)} \\ -1 & \text{when } (\alpha, \beta, \gamma) \text{ is an odd permutation of (1,2,3)} \\ 0 & \text{otherwise} \end{cases}$$

a) Using the summation convention (i.e. summarizing over twice appearing indices) the α-component of the vector product $\vec{a} \times \vec{b}$ can be expressed as $(\vec{a} \times \vec{b})_\alpha = \varepsilon_{\alpha\beta\gamma} a_\beta b_\gamma$. Show this for the component $\alpha = 1$.

b) Show the following identities:

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\mu\nu} &= \delta_{\beta\mu}\delta_{\gamma\nu} - \delta_{\beta\nu}\delta_{\gamma\mu}, \\ \varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta\nu} &= 2\delta_{\gamma\nu}, \\ \varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma} &= 3! \end{aligned}$$

c) Show that for a twice partially differentiable (vector-) function the following relations hold:

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \Delta\vec{a} \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) &= 0 \\ \vec{\nabla} \times (\vec{\nabla} f) &= 0 \end{aligned}$$

d) Show the following equation for a matrix M with components $M_{\alpha\beta}$:
$\det(M) = \varepsilon_{\alpha\beta\gamma} M_{\alpha 1} M_{\beta 2} M_{\gamma 3}$

Problem 1.12 *Delta function*

a) The δ-function can be represented as the limit, as $N \longrightarrow \infty$, of the sum

$$\sum_{k=-N}^{N} e^{2\pi i k x} \; .$$

Show the above sum is equal to

$$\frac{\sin[2\pi(N + \frac{1}{2})x]}{\sin(\pi x)}$$

for $x \neq 0$. Hint: Use Euler's formula and the identity

$$\sum_{k=0}^{2N} z^k = \frac{z^{2N+1} - 1}{z - 1} \ .$$

The 3-dimensional δ-function can be analogously represented as

$$\lim_{N\to\infty} \sum_{h=-N}^{N} \sum_{k=-N}^{N} \sum_{l=-N}^{N} e^{2\pi i(hx+ky+lz)} \ .$$

Show for $x, y, z \neq 0$, the above sum is equal to $f(x)f(y)f(z)$ where

$$f(x) = \frac{\sin[2\pi(N + \frac{1}{2})x]}{\sin(\pi x)} \ .$$

b) Evaluate the following integrals:

(i) $\int_0^3 x^3\ \delta(x+1)\ \mathrm{d}x$

(ii) $\int_{-1}^{1} 9x^2\ \delta(3x+1)\ \mathrm{d}x$

(iii) $\int_{V'} (r^2+2)\ \vec{\nabla} \cdot \left(\frac{\vec{e}_r}{r^2}\right)\ \mathrm{d}V$, where V' is a sphere of radius R centered at the origin, and $\vec{e}_r$ is the unit vector in the radial direction. Hint: Use the fact that $\vec{\nabla} \cdot \left(\frac{\vec{e}_r}{r^2}\right) = 4\pi\delta^3(\vec{r})$.

(iv) $\int_{V'} |\vec{r}-\vec{b}|^2\ \delta^3(5\vec{r})\ \mathrm{d}V$, where V' is a cube of side 2, centered on the origin, and $\vec{b} = 4\vec{e}_y + 3\vec{e}_z$.

Problem 1.13 *Dirac delta function*

Using Dirac delta function in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho_e(\vec{x})$.

a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

b) In cylindrical coordinates, a charge λ per unit length uniformly ditributed over a cylindrical surface of radius b.

c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

d) The same as part (c), but using spherical coordinates.

Problem 1.14 *Properties of the Dirac delta function*

The function $h(x)$ has only one simple root x_0. Explain the relation

$$\delta(h(x)) = \frac{1}{|h'(x_0)|}\delta(x - x_0).$$

Prove the following properties of the δ-function:

a) $x\delta(x) = 0$

b) $\varphi(x)\delta(x-a) = \varphi(a)\delta(x-a)$

c) $\displaystyle\int \mathrm{d}y\ \delta(x-y)\delta(y-z) = \delta(x-z)$

Problem 1.15 *Spherical harmonics*

a) The 'associated' Legendre polynomials P_l^m

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{\partial^{l+m}}{\partial x^{l+m}}(x^2-1)^l$$

can be calculated for $-l \leq m \leq l$. Verify that for $l = 0, ..., 3$ they fulfill the differential equation

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial P(x)}{\partial x}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0\,.$$

b) The spherical harmonics $Y_{lm}(\theta, \phi)$ are defined by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}\,.$$

Verify for $l, l' \leq 2$ the orthonormality condition

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \, Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}\,.$$

Electrostatics

Problem 2.16 *Gauß' law*

Using Gauß' law show that the normal derivative of the electric field on the surface of a charged conductor is

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are the principal radii of curvature on the surface of the conductor.

Problem 2.17 *Newton's theorem*

a) Prove the electrostatic analog of Newton's theorem:

   For a spherically symmetric charge (or mass, in the case of gravity) distribution $\rho(r)$, the radial component of the electric field, $E_r = \vec{E} \cdot \vec{r}/r$, is given by

$$E_r = \frac{Q(r)}{r^2} \qquad \text{with} \qquad Q(r) = 4\pi \int_0^r \rho(R) R^2 \mathrm{d}R\,,$$

   i. e. the same as if the charge in the sphere of radius R is located at the center of the sphere.

   Calculate also the associated electrostatic potential.

   Note that the Poisson equation in spherical coordinates reads

$$-4\pi\rho = \nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\varphi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\varphi}{\partial\phi^2}\,.$$

b) As an application of Newton's theorem, consider a charge-free spherical cavity concentric with the center of a spherically symmetric charge distribution. What is the electric force on a test charge inside this hole?

Problem 2.18 *Stable rest position in an electrostatic field*

Consider a time-independent electric field $\vec{E}(\vec{x})$ with the property that for a point particle with charge $q > 0$ the position $\vec{x} = 0$ is a stable rest position with a linear reset force. That is, for the force $\vec{F} = q\vec{E}$, the following properties hold:

- $\vec{F}$ has a zero of order one at $\vec{x} = 0$.
- In the vicinity of $\vec{x} = 0$ $\vec{F}$ forces the positive particle back to $\vec{x} = 0$. $\vec{F}(\vec{x})$ can be regarded as a linear function of $\vec{x}$ for small displacements.

The magnetic field is zero.

Show that the charge density $\rho(\vec{x})$, that creates the field $\vec{E}$, cannot have a zero at $\vec{x} = 0$. What sign does $\rho(0)$ have?

Hint: The relation $\vec{\nabla} \times \vec{E} = 0$ follows from the induction law. Use that fact to show that $A_{ij} := -\frac{\partial}{\partial x_i}E_i(\vec{x} = 0)$ is a symmetric matrix. Now choose the coordinate axes parallel to the direction of the principal axes of A_{ij} and calculate in this coordinate system the charge density $\rho(\vec{x})$.

Problem 2.19 *Cylindrical capacitor*

a) Using Gauß' law, calculate the capacitance of two concentric conducting cylinders of length L, when L is large compared to their radii a, b $(a < b)$. Apply the result to calculate the inner diameter of the outer conductor in an air-filled coaxial cable whose center is a cylindrical wire of diameter 1 mm and whose capacitance is $0.5 \times 10^{-6} \mu\text{F cm}^{-1}$.

b) For the cylindrical capacitor from part a) calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges Q and $-Q$ placed on the capacitor plates and the potential difference between the plates. Sketch the energy density of the capacitor's electrostatic field as a function of the appropriate linear coordinate.

c) Two long, parallel, cylindrical conductors of radii a_1 and a_2 are separated by a distance d, which is large compared with either radius. Show that the capacitance per unit of length is given approximately by

$$C \simeq \left[4 \ln \left(\frac{d}{a}\right)\right]^{-1},$$

where a is the geometrical mean of the two radii.

(i) What gauge wire (state the radius in millimetres) would be necessary to make a two-wire transmission line with a capacitance of 0.1 pF cm^{-1}, if the separation of the wires is 0.5 cm, 1.5 cm, and 5.0 cm?

d) Calculate the attractive force between the two conductors in a parallel cylindrical capacitor for:

(i) fixed charges on each conductor,

(ii) a fixed potential difference between the conductors.

Problem 2.20 *Conducting surfaces*

Prove the following theorem:

If a number of conducting surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

Problem 2.21 *Capacitances and self-induction coefficients*

Consider n conductors. The ith conductor carries the charge q_i and is at potential V_i. There is a linear relation between the charges and the potentials

$$q_i = \sum_{j=1}^{n} C_{ij} V_j$$

with constant quantities C_{ij}. The coefficients C_{ii} are called capacitances, and the C_{ij} $(i \neq j)$ are the self-induction coefficients.

a) Explain the relation above.

b) Calculate the C_{ij} for two concentric spherical shells of radii r_i and r_a. What relation exists between C_{ij} and the "capacitance" of the spherical capacitor?

Problem 2.22 *Electric dipole and quadrupole moments*

a) State the conditions under which the electric dipole and quadrupole moments are independent of the choice of the reference point.

b) Consider the following distribution of point charges, also shown in figure 1: a charge q_1 at $(a, 0, 0)$, a charge q_2 at $(0, a, 0)$, a charge q_3 at $(-a, 0, 0)$ and a charge q_4 at $(0, -a, 0)$. Calculate the dipole and quadrupole moments with respect to the reference point $\vec{r}_0 = 0$. Under what conditions is the dipole moment zero? Under what conditions is the quadrupole moment zero? Is it possible that the dipole and the quadrupole moments are zero at the same time?

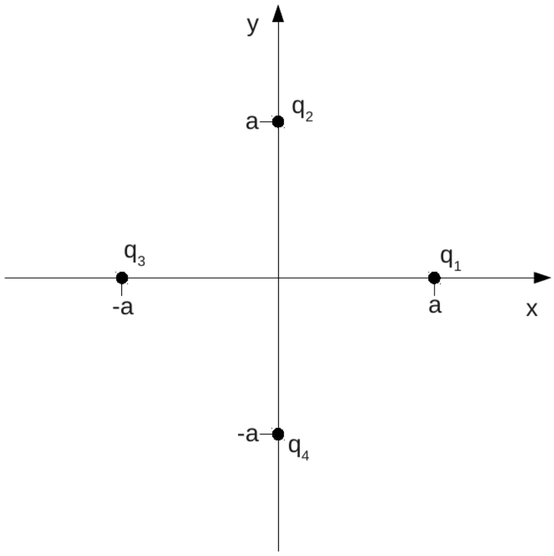


Fig. 1: Charge distribution of the exercise 5.1.b

Problem 2.23 *Green's reciprocation theorem*

Green's reciprocation theorem states that if Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, and Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi' \,\mathrm{d}V + \int_S \sigma\Phi' \,\mathrm{d}S = \int_V \rho'\Phi \,\mathrm{d}V + \int_S \sigma'\Phi \,\mathrm{d}S \,.$$

a) Two infinite, grounded, parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. Hint: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.

b) Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).
   (i) Write down the appropriate Green's function $G(\vec{x}, \vec{x}')$.
   (ii) Suppose that the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle.
   (1) Find an integral expression for the potential at the point P specified in terms of the cylindrical coordinates (ρ, ϕ, z).
   (2) Show that, along the axis of the circle $(\rho = 0)$, the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

(3) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2} \frac{z}{(\rho^2 + z^2)^{3/2}} \left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + ... \right] .$$

Verify that the results of parts (2) and (4) are consistent with each other in their common range of validity.

Problem 2.24 *Method of images I*

a) Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of inner radius a. Find
   (i) the potential inside the sphere;
   (ii) the induced surface-charge density;
   (iii) the magnitude and direction of the force acting on q.
   Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?
b) A conducting sphere is placed in a uniform external field. Calculate the potential Φ and the surface charge density σ.

Problem 2.25 *Method of images II*

a) A line charge with linear charge density τ is placed parallel to, and a distance R away from, the axis of a conducting cylinder of radius b held at fixed voltage such that the potential vanishes at infinity. Find
   (i) the magnitude and position of the image charge(s);
   (ii) the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x-axis), including the asymptotic form far from the cylinder;
   (iii) the induced surface-charge density, and plot it as a function of angle for $R/b = 2$, 4 in units of $\tau/(2\pi b)$;
   (iv) the force per unit length on the line charge.
b) (i) Use the method of images to show that the two-dimensional Dirichlet Green's function for the outside problem of a cylinder with a radius b is

$$G(\rho, \varphi; \rho', \varphi') = \ln \left\{ \frac{b^4 + \rho^2\rho'^2 - 2b^2\rho\rho' \cos(\varphi - \varphi')}{b^2[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')]} \right\} = \ln \left[\frac{(\rho^2 - b^2)(\rho'^2 - b^2) + b^2|\vec{\rho} - \vec{\rho'}|^2}{b^2|\vec{\rho} - \vec{\rho'}|^2} \right] ,$$

   where, $\vec{\rho}$ and $\vec{\rho'}$ are the coordinate vectors in a plane.
   (ii) Use the Green's function to check the result from **Problem 2.19**, part c).
   (iii) Are there any changes in case of the inside problem?

Problem 2.26 *Hollow conducting cylinder*

a) Two halves of a long hollow conducting cylinder of inner radius b are separated by small length-wise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1} \left(\frac{2b\rho}{b^2 - \rho^2} \cos\phi \right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.
b) Calculate the surface charge density on each half of the cylinder.

Problem 2.27 *Localized distribution of charge*

A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \frac{1}{64\pi} r^2 \mathrm{e}^{-r} \sin^2\theta$$

a) Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

b) Determine the potential explicitly at any point in space and show that near the origin, correct to r^2 inclusive,

$$\Phi(\vec{r}) \simeq \left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]$$

c) If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}\,\mathrm{m}^2$, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(\vec{r})$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\varepsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}$ m. Express your answer as a frequency by dividing by Planck's constant h.
The charge density in this problem is that for the $m = \pm 1$ states of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

Problem 2.28 *Dielectric medium*

Consider a charge e at the point $P = (x = 0, y = 0, z = -a)$, i.e. at a distance a from a boundary surface of two different, three-dimensional, homogenous, dielectric media with dielectric constants ε_1 and ε_2. The interface is defined by the plane $z = 0$.

a) φ_1 is the potential in the medium with ε_1 and φ_2 is the potential in the medium with ε_2. Derive a condition that connects φ_1 and φ_2 at the boundary layer by considering the tangential component of the electric field.

b) Derive another condition for φ_1 and φ_2 at the boundary layer by considering the normal component of the electric displacement.

c) Use the method of image charges and the boundary conditions of the previous questions to determine the potentials φ_1 and φ_2.
Hint: Choose an Ansatz for φ_1 with an image charge e' at an appropriate position in the medium 2 and for φ_2 an Ansatz with an image charge e'' at an appropriate position in the medium 1. After that, determine e' and e''.

d) Calculate the force acting on the charge e, if medium 1 is air ($\varepsilon_1 = 1$) and $\varepsilon_2 > 1$.

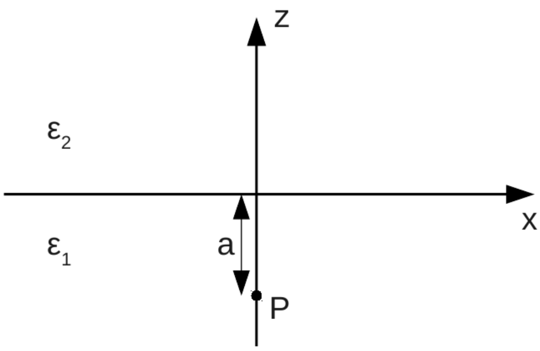
e) To what physical situation does the limiting case $\varepsilon_2 \to \infty$ correspond?

Problem 2.29 *Spherical and plane capacitors*

a) A conducting charged sphere of radius a has a total charge Q. Use Gauß' theorem to obtain the electric fields both inside and outside the sphere. Sketch the behavior of the fields as a function of the radius.

b) Using Gauß' law, calculate the capacitance of two concentric conducting spheres with radii a, b $(b > a)$.

c) For two large, flat sheets of area A, separated by a small distance d, and the capacitor geometry from part b) calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges Q and $-Q$ placed on the conductors and the potential difference between them.



(i) Sketch the energy density of the electrostatic field in each case as a function of the appropriate linear coordinate.

Problem 2.30 *Minimum of energy functional*

A volume V is bounded by a surface S consisting of several separate conducting surfaces S_i, each held at a fixed potential V_i. There is a possibility that one of the surfaces is at infinity. Let $\Psi(\vec{x})$ be an arbitrary, well-behaved function (i.e., it has no singularities) on V and S, such that it has the value V_i on the surface S_i. The energy functional is defined as

$$W[\Psi] = \frac{1}{8\pi} \int_V |\nabla\Psi|^2 \, \mathrm{d}V \, .$$

Prove the following theorem:
The non-negative functional $W[\Psi]$ is stationary and has an absolute minimum only if Ψ satisfies the Laplace equation on V and Ψ has values V_i on the surfaces S_i.

Problem 2.31 *Maximum of capacitance*

A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces S_i. One conuductor is held at *unit* potential and all the other conductors at zero potential.

a) Show that the capacitance of the one conductor is given by

$$C = \frac{1}{4\pi} \int_V |\nabla\Phi|^2 \mathrm{d}V$$

where $\Phi(\vec{x})$ is the solution for the potential.

b) Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi] = \frac{1}{4\pi} \int_V |\nabla\Psi|^2 \mathrm{d}V$$

where Ψ is any trial function satisfying the boundary conditions on the conductors. This is a variational principle for the capacitance that yields an *upper bound*.

Problem 2.32 *Interaction energy of two point charges*

Show that the interaction energy of two point charges is equal to $W = \dfrac{q_1 q_2}{|\vec{r_1} - \vec{r_2}|}$. The interaction energy is the total energy of the configuration minus the self-energy of the charges.

Problem 2.33 *The potential of the uniformly charged rod*

The charge q is distributed uniformly on the straight line of the length $2c$, as in Figure 1. Find the potential at each point in space. How does the potential look like very far from the rod (i.e., at

$l_1, l_2 \gg c$)? Also, find the equipotential surfaces. Hint: To find the equipotential surfaces, the elliptic coordinates $u = \frac{1}{2}(l_1 + l_2)$ and $v = \frac{1}{2}(l_1 - l_2)$ may be useful.

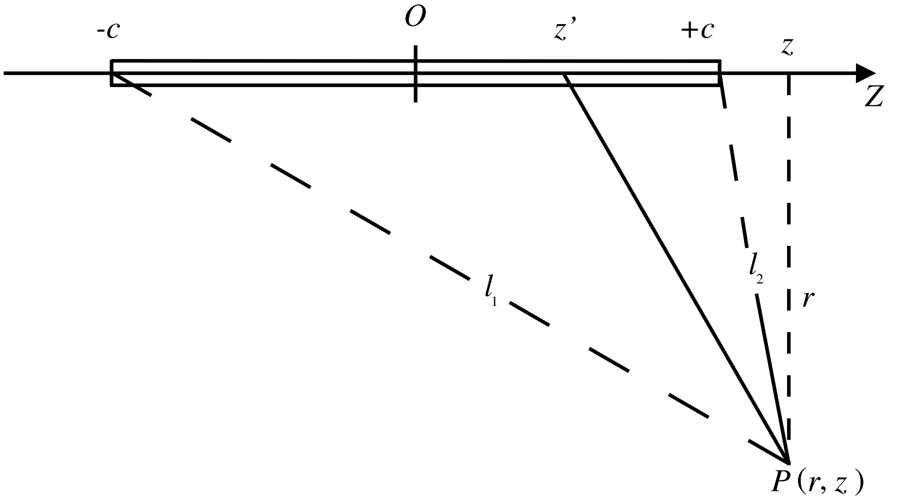


Fig. 1: Uniformly charged rod.

Problem 2.34 *Electric dipole and quadrupole moment*

Consider a group of point charges with charge distribution $\rho(\vec{r})$ that is symmetric with respect to an inversion at the $x = 0$ plane, i.e. the condition $\rho(x, y, z) = \rho(-x, y, z)$ holds.

a) Show that the x-component of the dipole moment with respect to the reference point $\vec{r}_0 = 0$ is zero.

b) What are the consequences for the components of the quadrupole moment?

Problem 2.35 *Plate capacitor*

The separation between the rectangular plates of a capacitor (see fig. 1) is $d - a$ on its lower edge, and $d + a$ on its upper edge, respectively. The width of the plates (along the parallel edges) is d, and the length is l. Neglect the edge effects and show that the capacitance of the capacitor is given by

$$C = \frac{d}{8\pi}\left[\arcsin\left(\frac{a}{l}\right)\right]^{-1} \ln\left(\frac{d+a}{d-a}\right) ,$$

and that for $a \ll d$, this capacitance approaches the capacitance of a parallel plate capacitor.

Problem 2.36 *Green's function*

The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi = 0$, $\phi = \beta$, and $\rho = a$, as indicated in Fig. 2.

Using separation of variables in polar coordinates, show that the Dirichlet Green's function can be written as

$$G(\rho, \phi; \rho', \phi') = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right) \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) .$$

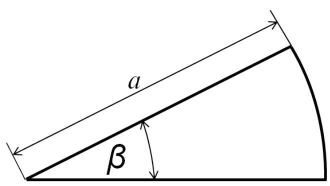


Fig. 2: Boundary surface of the problem.

Problem 2.37 *Hemispherical boss*

A large parallel plate capacitor is made up of two plane conducting sheets with separation D, one of which has a small hemispherical boss of radius a on its inner surface ($D \gg a$). The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is E_0.

a) Calculate the surface-charge densities at an arbitrary point on the plane and on the boss, and sketch their behavior as a function of distance (or angle).

b) Show that the total charge on the boss has the magnitude $3E_0a^2/4$.

c) If, instead of the other conducting sheet at a different potential, a point charge q is placed directly above the hemispherical boss at a distance d from its center, show that the charge induced on the boss is

$$q' = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right] .$$

Problem 2.38 *Hollow cube*

A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy?

c) Find the surface-charge density on the surface $z = a$.

Problem 2.39 *Hydrogen atom I*

The electronic charge distribution of a hydrogen atom in a p-orbital has the following form in spherical coordinates

$$\rho(\vec{R}) = -\frac{e}{64\pi a^3}\left(\frac{r}{a}\right)^2 \mathrm{e}^{-r/a}\sin^2\theta$$

where a is the Bohr radius and e is the elementary charge.

a) Calculate the multipole moment

$$q_{lm} = \int\limits_{\mathbb{R}^3} Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{r}) \mathrm{d}V'$$

and the multipole expansion

$$\bar{\Phi}(\vec{r}) := \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi)$$

Hint: $\sin^2\theta$ can be expressed as a linear combination of spherical harmonics.

b) How and why is $\bar{\Phi}$ different from the exact potential

$$\Phi(\vec{r}) = \int\limits_{\mathbb{R}^3} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \mathrm{d}V' \qquad ?$$

Hint: You may use the following relation

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi') \quad ,$$

where $r_>(r_<)$ denotes the larger (smaller) of the two radius vectors $\vec{r}$ and $\vec{r}'$.

c) What is the behaviour of $\Phi(\vec{r})$ for $r \ll a$?

Problem 2.40 *Hydrogen atom II*

Quantum mechanics reveals that the electron in a hydrogen atom should be described in terms of a wave function $\psi(\vec{r})$ (probability amplitude) giving rise to a smeared electron cloud corresponding to a charge density, $\rho_e(\vec{r}) = -e|\psi(\vec{r})|^2$. At the center of the atom, the proton is localized at a much smaller length scale, and the contribution to the charge density may be modeled as a point charge, $e\delta(\vec{r})$. Determine the (total) electrostatic potential φ

a) for the (1s orbital, K-shell) ground state of the hydrogen atom. Here the wave function is spherically symmetric
$$\psi(\vec{r}) = \frac{1}{\sqrt{\pi a^3}}\,\mathrm{e}^{-r/a}$$
where $a = \hbar^2/2me^2 = 0.529 \times 10^{-8}$cm denotes the Bohr radius.

b) for the spherically symmetric first excited state (2s orbital, L-shell)
$$\psi(\vec{r}) = \frac{1}{\sqrt{8\pi a^3}}\left(1 - \frac{r}{2a}\right)\mathrm{e}^{-r/2a}\,.$$

Problem 2.41 *Circular disc*

A thin, flat, conducting, circular disc of radius R is located in the x-y plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

a) show that for $r > R$ the potential is
$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta)$$

b) find the potential for $r < R$.

c) What is the capacitance of the disc?

Problem 2.42 *Multipole expansion*

The lth term in the multipole expansion of the potential
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$
is specified by the $(2l+1)$ multipole moments q_{lm}. On the other hand, the Cartesian multipole moments,
$$Q^{(l)}_{\alpha\beta\gamma} = \int \rho(\vec{x})x^\alpha y^\beta z^\gamma \mathrm{d}V$$
with α, β, γ nonnegative integers subject to the constraint $\alpha+\beta+\gamma = l$, are $(l+1)(l+2)/2$ in number. Thus, for $l > 1$ there are more Cartesian multipole moments than seem necessary to describe the term in the potential whose radial dependence is r^{-l-1}.

Show that while the q_{lm} transform under rotations as irreducible spherical tensors of rank l, the Cartesian multipole moments correspond to reducible spherical tensors of ranks $l, l-2, l-4, ..., l_{\text{min}}$, where $l_{\text{min}} = 0$ or 1 for l even or odd, respectively. Check that the number of different tensorial components adds up to the total number of Cartesian tensors. Why are only the q_{lm} needed in the multipole expansion?

Problem 2.43 *Cylindrical box*

A unit point charge is located at the point (ρ', ϕ', z') inside a grounded cylindrical box defined by the surfaces $z = 0$, $z = L$, $\rho = a$. Show that the potential inside the box can be expressed as

$$\Phi\left(\vec{x}, \vec{x}'\right) = \frac{4}{a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')} J_m(x_{mn}\rho/a) J_m(x_{mn}\rho'/a)}{x_{mn} J_{m+1}^2(x_{mn}) \sinh(x_{mn}L/a)} \sinh\left[\frac{x_{mn}}{a} z_<\right] \sinh\left[\frac{x_{mn}}{a}(L - z_>)\right] .$$

where $J_\nu(x)$ is a Bessel function of the first kind of order ν.

Problem 2.44 *Spherical cap*

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/(4\pi R^2)$, except for a spherical cap at the north pole, defined by the cone $\theta \leq \alpha$.

a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{2} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)] \frac{r^l}{R^{l+1}} P_l(\cos\theta)$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

b) Find the magnitude and the direction of the electric field at the origin.

c) Discuss the limiting forms of the potential (from part a) and electric field (from part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

Problem 2.45 *Hollow cylinder*

a) A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

b) For the cylinder in part (a) the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & ; \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ -V & ; \quad \frac{\pi}{2} < \phi < \frac{3\pi}{2} \end{cases} .$$

(i) Find the potential inside the cylinder.

(ii) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with the two-dimensional **Problem 2.26**.

Problem 2.46 *Quadrupole moment*

A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

a) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4} Q \left(\frac{\partial E_z}{\partial z}\right)_0 .$$

b) If it is known that $Q = 2 \times 10^{-24}$ cm^2 and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of e/a_0^3, where $a_0 = \hbar^2/me^2 = 0.529 \times 10^{-8}$ cm is the Bohr radius in hydrogen.

c) Nuclear-charge distributions can be approximated by a constant charge-density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153}($Z = 63$) has a quadrupole moment $Q = 2.5 \times 10^{-24}$ cm^2 and a mean radius

$$R = (a + b)/2 = 7 \times 10^{-13} \text{ cm},$$

determine the fractional difference in radius $(a - b)/R$.

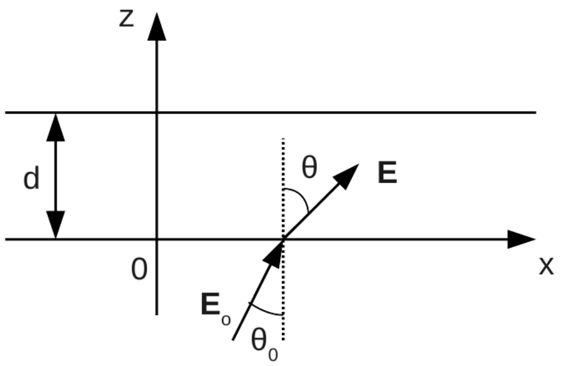
Problem 2.47 *Dielectric tensor*

Anisotropic dielectric media are described by a dielectric tensor ε_{ik}. So the following relation holds: $D_i = \sum_{k=x,y,z} \varepsilon_{ik} E_k \quad (i = x, y, z)$. A flat, in $x-$ and $y-$ direction, infinite plate of thickness d with the dielectric tensor ε_{ik} is placed in vacuum and in an external, homogeneous electric field $\vec{E}_0 = E_{0x}\vec{e}_x + E_{0z}\vec{e}_z$.

a) Calculate $\vec{E}(\vec{r}), \vec{D}(\vec{r})$ and $\vec{P}(\vec{r})$ within the plate.

b) Calculate the polarization charge density $\rho_p(\vec{r})$ and (for $z = 0$ and $z = d$) the surface polarization charge density $\sigma_p(\vec{r})$.

c) Calculate the angle of refraction θ of $\vec{E}$ at the interface $z = 0$ in terms of the incident angle θ_0.



Problem 2.48 *Spherical capacitor*

Consider two concentric, conductive hollow spheres with the radii $r_i < r_a$. The inner sphere has the charge Q, while the outer sphere has the charge $-Q$.

a) Calculate the $\vec{E}$- and the $\vec{D}$-fields, and the potential and the energy stored between the spheres.

b) Now a dielectrical (relative dielectric constant ε_r), thick-walled, hollow spherical shell (with an inner radius r_1 and an outside radius r_2) is placed concentrically between the other spheres (i.e. $r_i < r_1 < r_2 < r_a$). The charges on the hollow spheres stay the same.
Calculate again the $\vec{E}$- and the $\vec{D}$-fields, and the potential and the field energy.

Problem 2.49 *Debye-Hückel theory*

In an electrolyte solution ions of opposite charges can freely float agitated by thermal fluctuations. To simplify consider only one species of cations of charge $q_+ > 0$ and anions of charge $q_- < 0$ of respective number density n_+ and n_-. Charge neutrality requires $q_+n_+ + q_-n_- = 0$. The constitutive equation of the Debye-Hückel electrolyte relates the charge densities of the cations $\rho_+(\vec{x})$ and anions $\rho_-(\vec{x})$ to the electrostatic potential $\varphi(\vec{x})$ via

$$\rho_+(\vec{x}) = q_+n_+ \exp\left(-\frac{q_+\varphi(\vec{x})}{k_BT}\right), \qquad \rho_-(\vec{x}) = q_-n_- \exp\left(-\frac{q_-\varphi(\vec{x})}{k_BT}\right).$$

Here T denotes the temperature and k_B is Boltzmann's constant.

a) Considering external charges $\rho^{\text{ext}}(\vec{x})$ in addition to the induced ones, $\rho^{\text{ind}}(\vec{x}) = \rho_+(\vec{x}) + \rho_-(\vec{x})$, formulate the Poisson equation. Linearize the exponentials in $\varphi(\vec{x})$ and show that

$$\nabla^2\varphi(\vec{x}) - \frac{1}{\lambda^2}\varphi(\vec{x}) = -4\pi\rho^{\text{ext}}(\vec{x}),$$

holds. Relate the Debye-Hückel *screening length* λ to the number densities $n_\pm$.

b) Determine the electrostatic potential within the linearized theory for a point-like test charge $\rho^{\text{ext}}(\vec{x}) = Q\delta(\vec{x})$. Use the spherical symmetry of the problem and determine the solution of the differential equation for $r = |\vec{x}| > 0$ that vanishes for $r \to \infty$. Discuss the physical consequences of the result.

Hint: The substitution $\varphi(r) = u(r)/r$ simplifies the homogeneous differential equation. The constant of integration may be determined by matching to the Coulomb solution in vacuum close to the test charge.

c) Equivalently you may evaluate the Green function $G(\vec{x}, \vec{y})$ defined via

$$\left(\frac{1}{\lambda^2} - \nabla^2\right) G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}).$$

Show that

$$G(\vec{k}) = \int \mathrm{d}^3\vec{x}\ \mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} G(\vec{x}, \vec{y}).$$

satisfies an algebraic equation and perform the inverse Fourier transform; apply the residue theorem to perform the integration.

d) *Not compulsory but interesting*
Consider a small spherical colloid of radius R suspended in the electrolyte. The colloid carries a charge Ze homogeneously distributed along the surface. There are no further charges inside of the colloid. Since the electrolyte cannot penetrate the colloidal particles, the usual Poisson equation holds in the inner region. Determine the electrostatic potential and compare your result to the point-like test charge.

Magnetostatics

Problem 3.50 *Rotating hollow sphere*

A charge Q is uniformly distributed over a hollow sphere of radius R and negligible thickness. The sphere rotates around an axis through its center with constant angular velocity ω.

a) Show that with the rotation of the hollow sphere one can associate a stationary current density

$$\vec{J}(\vec{r}\,) = \frac{Q}{4\pi R^2}\,\delta(|\vec{r}\,| - R)\,\vec{\omega}\times\vec{r}$$

where $\vec{\omega} = \omega\vec{n}$, and $\vec{n}$ is the unit vector in the direction of the rotation axis.

b) Calculate the vector potential $\vec{A}(\vec{r}\,)$ and the magnetic field $\vec{B}(\vec{r}\,)$ inside and outside the sphere.

c) Calculate the magnetic moment $\vec{m}$ of the sphere as well as $\vec{B}(\vec{r}\,)$ with the dipole approximation. Show that in the range $|\vec{r}\,| > R$ the exact field matches with the dipole approximation.

Problem 3.51 *Solenoid deformation*

A long, flexible, cylindrical solenoid (spring) with a radius a, a length L and a negligible mass is composed of N turns of a wire. The solenoid is fixed at its upper end and a weight of mass w is hung at its lower end. Find the current through the solenoid so that the weight induces no deformation. Ignore the edge effects at the ends of the solenoid.

Problem 3.52 *Dipole fields*

a) Calculate the field of a magnetic moment $\vec{m}$, and a dipole moment $\vec{p}$ and pay attention to the values of the fields at the position of the dipole.

b) Check the Maxwell equations $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\cdot\vec{E} = 4\pi\rho(\vec{r}\,)$.

Problem 3.53 *Polarization and magnetization*

For a static polarization field $\vec{P}(\vec{r})$ that vanishes sufficiently rapidly at infinity, an electrostatic potential $\varphi(\vec{r})$ is given by

$$\varphi(\vec{r}) = -\vec{\nabla}_r\cdot\int\frac{\vec{P}(\vec{R})}{|\vec{r}-\vec{R}|}\,\mathrm{d}^3\vec{R}\,.$$

a) Argue that this expression indeed represents a solution of Poisson's equation,

$$-\nabla^2\varphi = 4\pi\rho^{(\mathrm{ind})} = -4\pi\vec{\nabla}\vec{P}.$$

Use the preceding result to calculate the electrostatic potential $\varphi(\vec{r})$ corresponding to a sphere of homogeneous polarization, $\vec{P} = \mathit{const}$. Determine also the electric field and sketch the field lines. Similarly, for a static magnetization field $\vec{M}(\vec{r})$ a solution of the magnetostatic problem is provided in terms of the vector potential

$$\vec{A}(\vec{r}) = \vec{\nabla}_r\times\int\frac{\vec{M}(\vec{R})}{|\vec{r}-\vec{R}|}\,\mathrm{d}^3\vec{R}\,.$$

Corroborate again that the preceding formula constitutes a solution of

$$-\nabla^2\vec{A} = 4\pi\vec{j}^{(\mathrm{ind})}/c = 4\pi\vec{\nabla}\times\vec{M}.$$

d) Determine a vector potential for a homogeneously magnetized sphere, $\vec{M} = \mathit{const}$, and calculate the magnetic field.

e) Argue that the magnetic fields arising due to a static magnetization field $\vec{M}$ can be expressed in terms of a scalar magnetostatic potential $\varphi_M(\vec{r})$ by $\vec{H} = -\vec{\nabla}\varphi_M$. Determine the field equation for φ_M that contains $\vec{M}$ as source terms. Compare the polarized with the magnetized sphere.

Problem 3.54 *Superconductor*

The constitutive equation of a type-I superconductor relates the supercurrent density $\vec{J}_s$ directly to the vector potential $\vec{A}$ via the second London equation,

$$\vec{J}_s(\vec{x}) = -\frac{n_s e^2}{mc}\vec{A}(\vec{x})\,.$$

Here m and $-e$ denote the mass and charge of the supercurrent carrier, and n_s abbreviates their number density.

(a) Use Maxwell's equations to show that in the static case the magnetic field fulfills the field equation

$$\nabla^2\vec{B}(\vec{x}) - \frac{1}{\lambda_L^2}\vec{B}(\vec{x}) = 0\,,$$

and determine the *London penetration depth* λ_L. Conclude that no homogeneous magnetic field can exist in the bulk of a superconductor.

(b) Consider the boundary $z = 0$ between a superconductor ($z > 0$) and vacuum ($z < 0$). A magnetic field $\vec{B}$ is applied parallel to the boundary ($z < 0$). Solve for the magnetic field inside of the superconductor.

(c) Show that the field equation can be obtained by minimizing the total energy $U = U_{\text{matter}} + U_{\text{field}}$ by varying with respect to the vector potential, $\vec{A}(\vec{x}) \to \vec{A}(\vec{x}) + \delta\vec{A}(\vec{x})$. Here the variation of the matter and field energy follows from

$$\delta U_{\text{matter}} = -\frac{1}{c}\int \mathrm{d}^3\vec{x}\,\vec{J}_s(\vec{x})\cdot\delta\vec{A}(\vec{x}) \qquad \text{and} \qquad \delta U_{\text{field}} = \frac{1}{4\pi}\int \mathrm{d}^3\vec{x}\,\vec{B}(\vec{x})\cdot\delta\vec{B}(\vec{x})\,.$$

(d) The supercurrent $J_s(\vec{x})$ and the magnetic field $\vec{B}(\vec{x})$ have to be eliminated in favor of $\vec{A}(\vec{x})$ to perform the variation.

Problem 3.55 *Vector potential*

The vector potential $\vec{A}$ corresponding to a solenoidal field $\vec{B}$, div $\vec{B} = 0$, $\vec{B} = \vec{\nabla}\times\vec{A}$, may be obtained by evaluating the line integral (Poincaré's lemma)

$$\vec{A}(\vec{x}) = -\int_0^1 u(\vec{x} - \vec{x}_0)\times\vec{B}(\vec{x}(u))\,\mathrm{d}u \qquad (*)$$

for straight lines $\vec{x}(u) = \vec{x}_0 + u(\vec{x} - \vec{x}_0)$.

a) Recall Ampère's law of magnetostatics, $\vec{\nabla}\times\vec{B} = 4\pi\vec{j}/c$. Thus in the case of a current-free region, $\vec{j} = 0$, a scalar magnetostatic potential φ_M may by introduced, $\vec{B} = -\vec{\nabla}\varphi_M$, where $\nabla^2\varphi_M = 0$. Employ Poincaré's lemma to determine a vector potential $\vec{A}$ of a magnetic octopole field corresponding to the potential

$$\varphi_M(\vec{x}) = z^3 - \frac{3}{2}(x^2 + y^2)z.$$

b) Evaluate the curl of the integral representation $(*)$ for $\vec{A}$ to prove that indeed $\vec{B} = \vec{\nabla}\times\vec{A}$ provided div $\vec{B} = 0$.

Problem 3.56 *Magnetostatics*

a) Consider an infinitely long, conducting solid cylinder of radius R and constant current density $\vec{J}_0$. Calculate the vector potential $\vec{A}$ and the magnetic field strength $\vec{B}$ inside and outside of the cylinder using the Poisson equation for the vector potential.

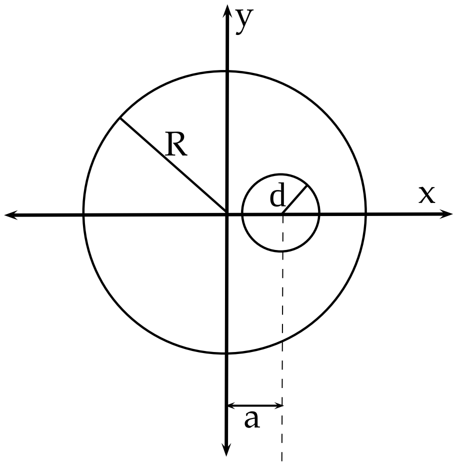
$$\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}\,.$$

Note: Use the symmetry of the problem.

b) Check the result for the magnetic field $\vec{B}$ in a) by using the Stokes equation (Ampère's law).

Problem 3.57 *Cylindrical wire*

Consider a cylindrical wire parallel to the z-axis $\vec{e}_z$ with a cylindrical hole parallel to the axis of the cylinder and shifted from the origin (see fig. below). Inside the wire material flows a constant current density $\vec{J}(\vec{r}\,) = J\vec{e}_z$. Calculate the magnetic field $\vec{B}(\vec{r}\,)$ at one point on the x-axis for $x > R$ and $a - d < x < a + d$. Express $\vec{B}(\vec{r}\,)$ in each case in terms of $\vec{J}(\vec{r}\,)$. Hint: First calculate the magnetic field for a full cylinder, i.e. without the hole, and then use the superposition of the currents.



Cylindrical wire with a hole

Electrodynamics

Problem 4.58 *Momentum conservation law*

Defining the symmetric tensor field (Maxwell stress tensor)

$$T_{ik}(\vec{x},t) = \frac{1}{4\pi}\left[\frac{1}{2}\delta_{ik}(\vec{E}^2+\vec{B}^2) - E_iE_k - B_iB_k\right] \qquad (i,k=1,2,3),$$

show that Maxwell's equations imply a local balance law for the momentum density,

$$\frac{1}{c^2}\partial_t S_i + \nabla_k T_{ik} = -F_i\,,$$

where $\vec{S} = (c/4\pi)\vec{E}\times\vec{B}$ denotes the Poynting vector. Determine the mechanical force density $\vec{F}$.

Hint: The following vector identity may prove useful,

$$\left[\vec{V}\times(\vec{\nabla}\times\vec{V})\right]_i = -\nabla_k\left(V_iV_k - \frac{1}{2}\delta_{ik}\vec{V}^2\right) + V_i\,\mathrm{div}\,\vec{V}\,.$$

Problem 4.59 *Charged particle with $\vec{E}\times\vec{B}$ drift*

A point particle with charge e is moving in a static electromagnetic field, i. e. the equation of motion reads,

$$m\dot{\vec{v}} = e\left(\vec{E} + \frac{1}{c}\,\vec{v}\times\vec{B}\right).$$

Calculate the velocity $\vec{v}(t)$ and the trajectory $\vec{r}(t)$ of the particle. Show that the fields impose a constant drift term on the velocity being proportional to $\vec{E}\times\vec{B}$. For the case $E_z = 0$, give a sketch of the trajectory of the particle and discuss the different cases of this cycloid motion. Finally, show explicitly that the energy gain of the particle vanishes on average.
Choose the coordinate frame such that the magnetic field defines the z-axis, $\vec{B} = B\hat{e}_z$; then, the motion along the z-axis separates. The remaining two coupled differential equations may be solved by introducing a complex velocity, $\zeta := v_x + \mathrm{i}v_y$, and a complex field, $\mathcal{E} := E_x + \mathrm{i}E_y$.

Problem 4.60 *Angular momentum conservation law*

The angular momentum density of the electromagnetic field is defined by the antisymmetric tensor field

$$L_{ij}(\vec{x},t) = \frac{1}{c^2}(x_iS_j - x_jS_i)\,,$$

where $\vec{S}$ denotes the Poynting vector.

a) Employ the momentum balance law to construct a local balance law for the angular momentum density of the form

$$\partial_t L_{ij} + \nabla_k M_{ijk} = -D_{ij}\,.$$

Determine the angular moment current tensor M_{ijk} as well as the mechanical torque tensor D_{ij}. Rewrite the balance law in terms of the pseudo-vector field

$$L_i(\vec{x},t) = \frac{1}{2}\varepsilon_{ijk}L_{jk}\,,$$

and suitable M_{ik} and D_i.

b) Formulate the angular momentum conservation law in integral form, for $\mathcal{L}_i = \int_V L_i\,\mathrm{d}V$.

c) Demonstrate that in the gauge $\varphi = 0$, the angular momentum of the field can be decomposed, $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_B$, in a 'spin' part

$$\mathcal{L}_S = \frac{1}{4\pi c^2}\int_V \vec{A}\times\dot{\vec{A}}\,\mathrm{d}V\,,$$

and an 'orbital' part $\mathcal{L}_B$ that depends explicitly on the point of reference of the coordinate system.

Problem 4.61 *Center-of-energy*

Consider the vector field $\vec{K}(\vec{x},t) := \vec{x}u(\vec{x},t) - t\vec{S}(\vec{x},t)$, where $u = (\vec{E}^2 + \vec{B}^2)/8\pi$ denotes the energy density of the electro-magnetic field and $\vec{S} = (c/4\pi)\vec{E} \times \vec{B}$ the Poynting vector.

a) Show that $\vec{K}(\vec{x},t)$ fulfills a generalized continuity equation

$$\partial_t K_i + \nabla_j N_{ik} = R_i$$

such that the source $\vec{R}(\vec{x},t)$ vanishes in the absence of charges and currents. Find a suitable current density $N_{ik}(\vec{x},t)$ and determine $\vec{R}(\vec{x},t)$.

b) Formulate a corresponding integral form and interpret the balance equation in the source-free case.

Problem 4.62 *Poincaré gauge*

Show that an arbitrary electromagnetic field, defined by the electric field $\vec{E}(\vec{r},t)$ and the magnetic field $\vec{B}(\vec{r},t)$, can be described by the electromagnetic potentials $\Phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$

$$\begin{aligned}\Phi(\vec{r},t) &= -\,\vec{r}\cdot\int_0^1 \mathrm{d}\lambda \vec{E}(\lambda\vec{r},t)\\ \vec{A}(\vec{r},t) &= \int_0^1 \mathrm{d}\lambda\,\lambda\vec{B}(\lambda\vec{r},t)\times\vec{r}\,.\end{aligned}$$

This choice of the electromagnetic potentials represents the so-called Poincaré's gauge.

Problem 4.63 *Retarded potentials*

Show that the retarded potentials

$$\Phi(\vec{r},t) = \int \frac{\rho(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c})}{|\,\vec{r}-\vec{r}'\,|}\mathrm{d}V', \tag{1}$$

and

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c})}{|\,\vec{r}-\vec{r}'\,|}\mathrm{d}V', \tag{2}$$

satisfy

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0, \tag{3}$$

under the condition that the above integrals converge.

Problem 4.64 *LRC circuit*

a) Relying on Kirchhoff's laws, argue that the charge on the capacitor $Q(t)$ (Fig. 1) fulfills the second order differential equation

$$L\ddot{Q}(t) + R\dot{Q}(t) + \frac{1}{C}Q(t) = V(t)\,.$$

b) First consider $V(t) \equiv 0$. Determine the charge $Q(t)$ for the initial conditions $Q(t=0) = Q_0$, $\dot{Q}(t=0) = I_0$. Separate the cases of small and large damping.

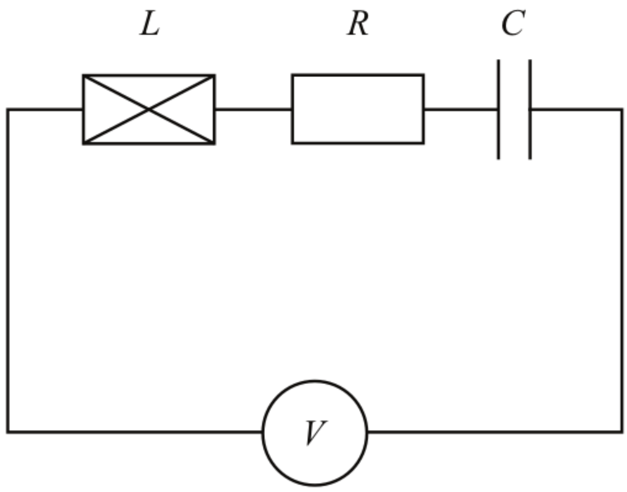


Fig. 3: An LRC circuit.

c) Determine the response $Q(t)$ due to a short voltage pulse, $V(t) = \Phi\delta(t)$, assuming $Q(t < 0) \equiv 0$. To obtain the solution, rewrite the second order differential equation into a set of two coupled first order equations

$$\dot{Q}(t) - I(t) = 0\,, \qquad L\dot{I}(t) + RI(t) + \frac{1}{C}Q(t) = \Phi\delta(t)\,.$$

For times $t > 0$, one recovers the homogeneous equations with corresponding solutions derived above. By integrating over a small time interval, $-\varepsilon \le t \le \varepsilon$ ($\varepsilon \downarrow 0$) show that the pulse induces a current $I(t = 0^+) = \Phi/L$, but no charge, $Q(t = 0^+) = 0$, which serve as initial conditions.

d) A general time-dependent applied voltage may be represented as a linear superposition of pulses, $V(t) = \int \mathrm{d}\bar{t}\, V(\bar{t})\delta(t-\bar{t})$, where $\delta(t)$ denotes Dirac's delta function. Show that the general solution for the charge in the limit of weak damping is given by

$$Q(t) = \int_{-\infty}^{\infty} \mathrm{d}\bar{t}\, \chi(t - \bar{t})\, V(\bar{t}) \qquad \text{where} \qquad \chi(t) = \Theta(t)\mathrm{e}^{-\gamma t/2}\frac{\sin(\omega_r t)}{\omega_r L}\,,$$

with suitably chosen γ and ω_r, and $\Theta(t)$ denotes the Heaviside step function,

$$\Theta(t) = \begin{cases} 1 & \text{for } t \ge 0, \\ 0 & \text{else.} \end{cases}$$

You may either rely on the superposition principle and use the previous considerations or prove the relation by direct substitution.

e) Perform a Fourier transform of $\chi(t)$, convention $\chi_\omega = \int_{-\infty}^{\infty} \chi(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t$, and determine the frequency-dependent complex susceptibility χ_ω. By the convolution theorem, the charge response in the frequency domain is related to the external driving by $Q_\omega = \chi_\omega V_\omega$. Show that χ_ω may be obtained much easier by a Fourier transform of the differential equation. Determine the real part χ'_ω and imaginary part χ''_ω of $\chi_\omega = \chi'_\omega + \mathrm{i}\chi''_\omega$ and show that

$$\chi'_\omega = \int_{-\infty}^{\infty} \chi(t)\cos(\omega t)\mathrm{d}t\,, \qquad \chi''_\omega = \int_{-\infty}^{\infty} \chi(t)\sin(\omega t)\mathrm{d}t\,.$$

Sketch χ'_ω and χ''_ω as a function of frequency ω.

f) Determine the current $I(t)$ for harmonic driving, $V(t) = \mathrm{Re}(V_\omega \mathrm{e}^{-\mathrm{i}\omega t})$. Calculate the dissipated power of the circuit and state its connection to χ''_ω.

Problem 4.65 *Polarizable Medium*

Consider the following constitutive equation for a polarizable medium

$$\partial_t^2 \vec{P}(\vec{x},t) + \frac{1}{\tau}\partial_t \vec{P}(\vec{x},t) + \omega_0^2 \vec{P}(\vec{x},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{x},t)\,;$$

here $\omega_p^2 = 4\pi n e^2/m$ denotes the plasma frequency, ω_0^2 the resonance frequency, and $\tau > 0$ is a characteristic damping time. The change of polarization is considered slow and possibly induced magnetic fields shall be neglected.

(a) Employing the constitutive equation, evaluate the time derivative $\dot{u}_M(\vec{x},t)$ of the energy density of matter

$$u_M(\vec{x},t) = \frac{2\pi}{\omega_p^2}\left(\vec{j}^{(\mathrm{ind})}(\vec{x},t)^2 + \omega_0^2 \vec{P}(\vec{x},t)^2\right)\,,$$

and interpret the terms contributing to $\dot{u}_M(\vec{x},t)$.

(b) Consider the total energy density $u = u_M + u_F$, with the usual field energy density $u_F = (1/8\pi)[\vec{E}^2 + \vec{B}^2]$, derive a local balance equation

$$\partial_t u(\vec{x},t) + \mathrm{div}\vec{S}(\vec{x},t) = q(\vec{x},t) \quad \text{where} \quad \vec{S}(\vec{x},t) = \frac{c}{4\pi}\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\,,$$

and determine the source term $q(\vec{x},t)$.

Problem 4.66 *Nuclear Magnetic Resonance*

Nuclear magnetic resonance spectroscopy uses the magnetic moment of the nuclei of certain atoms to study physical, chemical, and biological properties of matter. The magnetization $\vec{M}$ due to the spin of the nuclei obeys the *Bloch* equations

$$\dot{\vec{M}}(t) = \gamma\vec{M}(t)\times\vec{H}(t) - \frac{1}{T_1}\left[\vec{M}(t) - \vec{M}_0\right].$$

Here the gyromagnetic ratio γ determines the frequency of the *Larmor precession.* The second term is a phenomenological damping term introducing a characteristic (energy) relaxation time T_1. Consider a strong d.c. field $\vec{H}_0$ aligning the magnetization $\vec{M}(t) = \vec{M}_0 \parallel \vec{H}_0$ in the static case. A small time-dependent field $\delta H_\perp(t)$ is applied in addition to the d.c. field $\vec{H}_0$. The probing field $\delta\vec{H}_\perp(t)$ acts perpendicularly to $\vec{H}_0$ at all times.

(a) Derive a constitutive equation for the induced magnetization $\delta\vec{M}(t) = \vec{M}(t) - \vec{M}_0$ to linear order in $\delta\vec{H}_\perp(t)$. Decompose the response into a component parallel and perpendicular to the static external field, $\delta\vec{M}(t) = \delta\vec{M}_\parallel(t) + \delta\vec{M}_\perp(t)$, and show that they fulfill

$$\delta\dot{\vec{M}}_\parallel(t) + \frac{1}{T_1}\delta\vec{M}_\parallel(t) = 0\,, \qquad \delta\dot{\vec{M}}_\perp(t) - \gamma\delta\vec{M}_\perp(t)\times\vec{H}_0 + \frac{1}{T_1}\delta\vec{M}_\perp(t) = \gamma\vec{M}_0\times\delta\vec{H}_\perp(t)\,.$$

(b) Discuss the free decay of the induced magnetization $\delta\vec{M}(t)$ in the absence of external driving, i.e., $\delta\vec{H}_\perp(t)\equiv 0$, for arbitrary initial condition $\delta\vec{M}(t=0)$.

Hint: It is favorable to complexify the transverse magnetization $\delta\vec{M}_\perp(t)$.

(c) Derive the steady state response for a probing field rotating perpendicularly to the aligning field $\vec{H}_0$ at constant angular frequency, $\delta\vec{H}_\perp(t) = \delta H_\perp^\omega(\cos\omega t, -\sin\omega t, 0)$. Here the z-axis has been chosen parallel to $\vec{H}_0$.

Problem 4.67 *Penning trap*

Consider the motion of a particle that has a charge q and mass m in a constant uniform magnetic field $\vec{B} = B\hat{e}_z$ and an electric quadrupole potential ($U_0 > 0$)

$$\varphi(\vec{x}) = -\frac{U_0}{2r_0^2}(x^2 + y^2 - 2z^2)\,, \qquad \vec{x} = (x, y, z)\,.$$

a) Show that the non-relativistic equation of motion for the particle in the x–y plane for the case $U_0 = 0$ leads to oscillatory motion. Determine the cyclotron frequency ω_c characterizing the oscillation. It is favorable to introduce a complex variable $\xi := x + \mathrm{i}y$.

b) Determine the electric field $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$ and verify that $\vec{E}$ is solenoidal, i.e., $\vec{\nabla} \cdot \vec{E}(\vec{x}) = 0$.

c) Show that the magnetic field does not couple to the motion along the z-direction, and determine the characteristic frequency ω_z for the corresponding harmonic oscillations in the quadrupole field.

d) Solve the complete equations of motion in the x–y plane and show that the general solution is a superposition of two oscillatory motions with a perturbed cyclotron frequency ω_c' and the *magnetron* frequency ω_M. Provide conditions such that the orbits are stable. Discuss the case $\omega_z \ll \omega_c$ in particular.

Problem 4.68 *Sound waves in fluid*

The macroscopic properties of a fluid are characterized in terms of a few fields, e.g., the mass density $\rho(\vec{r}, t)$, the mass current density $\vec{j}(\vec{r}, t)$, the fluid velocity $\vec{v}(\vec{r}, t)$, and the pressure $p(\vec{r}, t)$. Euler's equations specify the field equations; the first set encodes the conservation of mass and momentum,

$$\partial_t \rho + \nabla_k\, j_k = 0\,, \quad \partial_t j_k + \nabla_l\, \Pi_{kl} = 0\,. \tag{$*$}$$

Note that both equations above are basically equations of continuity. The mass current density is connected to the fluid velocity by $\vec{j}(\vec{r}, t) = \rho(\vec{r}, t)\vec{v}(\vec{r}, t)$, and Π_{kl} denotes the momentum current tensor

$$\Pi_{kl} = \rho v_k v_l - \sigma_{kl} = \rho v_k v_l + p\delta_{kl}\,, \tag{$**$}$$

which closes the equations. The term $\rho v_k v_l$ is the contribution to the momentum current by the inertia of the flow (the terms responsible for turbulence). The quantity $\sigma_{kl} = -p\delta_{kl} + \sigma'_{kl}$ is known as the stress tensor. p denotes the pressure while σ'_{kl} encompasses the (bulk and shear) viscous forces, i.e. dissipative processes, which are neglected in Euler's equations, $\sigma'_{kl} = 0$.

a) Demonstrate that

$$\rho(\vec{r}, t) = \rho_0 = const, \quad \vec{v}(\vec{r}, t) = 0, \quad \text{and} \quad p(\vec{r}, t) = p_0 = const$$

constitutes a solution of the field equations. Show that the linearized field equations for small perturbations $\delta\rho = \rho - \rho_0$, $\vec{v}$, and $\delta p = p - p_0$ to this reference state read

$$\partial_t \delta\rho + \rho_0 \nabla_k v_k = 0\,, \qquad \rho_0 \partial_t v_k = -\nabla_k \delta p\,.$$

Introduce the *isothermal compressibility* κ_T that reflects the pressure increase due to compression at constant temperature to linear order, $\delta p = \delta\rho/\rho_0\kappa_T$.

b) Derive a local conservation law, $\partial_t u + \vec{\nabla} \cdot \vec{S} = 0$, for the energy density

$$u(\vec{r}, t) = \frac{\rho_0}{2}\vec{v}(\vec{r}, t)^2 + \frac{A}{2}\delta\rho(\vec{r}, t)^2$$

for suitably chosen A relying on the approximations introduced so far. Determine the energy current density $\vec{S}(\vec{x}, t)$.

c) Show that the linearized field equations allow for monochromatic longitudinal waves in $\vec{v}$ and scalar waves in $\delta\rho(\vec{r}, t)$.

Problem 5.69 *Rotating Dipole*

A dipole of a constant magnitude rotates in a plane around a fixed point with the angular velocity ω. Calculate the radiated electric and the magnetic field, the polarity, the angular distribution of radiation averaged over a period of dipole motion $\langle \mathrm{d}I/\mathrm{d}\Omega \rangle$, and the radiated power.

Problem 5.70 *Paraxial beams*

Consider a monochromatic beam of angular frequency $\omega \equiv kc$ propagating essentially along the positive z-direction.
a) Argue that the components of the electric field allow for a representation as

$$E(\vec{x}_\perp, z; t) = \mathrm{e}^{-\mathrm{i}\omega t} \int \frac{\mathrm{d}^2\vec{k}_\perp}{(2\pi)^2} a(\vec{k}_\perp) \exp(\mathrm{i}\vec{k}_\perp \vec{x}_\perp + \mathrm{i}k_\| z)\,, \tag{4}$$

where $k_\| = (k^2 - \vec{k}_\perp^2)^{1/2}$ is to be eliminated in favor of $\vec{k}_\perp$. The complex amplitude $a(\vec{k}_\perp)$ is assumed to contribute only for $|\vec{k}_\perp| \ll k$.
b) Expand the square root $k_\| \doteq k(1 - \vec{k}_\perp^2/2k^2)$ to leading order in $\vec{k}_\perp/k$ and show that the field assumes the following form

$$E(\vec{x}_\perp, z; t) = \mathrm{e}^{\mathrm{i}kz - \mathrm{i}\omega t} \mathcal{E}(\vec{x}_\perp, z)\,,$$

where the *envelope function* $\mathcal{E}$ is slowly varying along z on the scale of a wavelength, $\partial_z \mathcal{E} \ll k\mathcal{E}$. Relate the envelope to the amplitudes $a(\vec{k}_\perp)$. Show that the envelope satisfies the Schrödinger-like field equation

$$\mathrm{i}\partial_z \mathcal{E} = -\frac{1}{2k} \nabla_\perp^2 \mathcal{E}\,.$$

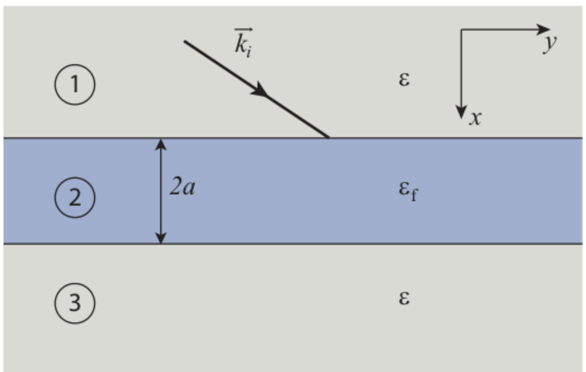
In particular, the field equation is first order in the z-direction.
c) Evaluate the electric field $E(\vec{x}_\perp, z; t)$ for a gaussian amplitude function

$$a(\vec{k}_\perp) \propto \exp(-\frac{1}{4} w_0^2 \vec{k}_\perp^2)\,, \qquad w_0 > 0\,,$$

and show that the intensity $I \propto |E|^2$ exhibits a gaussian profile in the perpendicular direction $\vec{x}_\perp$ and a width that depends on z. Where is the width minimal?

Problem 5.71 *Tunnel effect*

Consider a thin film of thickness $2a$ characterized by a dielectric constant ε_f dividing the three-dimensional space (with dielectric constant ε), see figure. A monochromatic electromagnetic plane wave is incident on the film; the coordinate system is chosen such that the incident wave vector reads $\vec{k}_\mathrm{i} = (k, k_\|, 0)$. Discuss the case of a polarization of the incident electric field parallel to the interfaces, $\vec{E}_\mathrm{i} = (0, 0, E_\mathrm{i})$, i.e., out of the drawing plane.



a) Determine the dispersion relation $\omega = \omega(\vec{k})$ separately in each region.

b) Argue that the polarization of the electric field is parallel to the interface in all three regions.

c) Since the tangential component of the electric field is continuous at the interfaces, the spatio-temporal modulations at the interface are identical. Justify the following Ansatz for the electric field

$$E_z(\vec{x},t) = \mathrm{e}^{\mathrm{i}k_\parallel y - \mathrm{i}\omega t} \begin{cases} E_\mathrm{i}\mathrm{e}^{\mathrm{i}kx} + E_\mathrm{r}\mathrm{e}^{-\mathrm{i}kx} & \text{for } x < -a, \\ E_+\mathrm{e}^{\mathrm{i}qx} + E_-\mathrm{e}^{-\mathrm{i}qx} & \text{for } -a < x < a, \\ E_\mathrm{t}\mathrm{e}^{\mathrm{i}kx} & \text{for } x > a\,, \end{cases}$$

and interpret the individual terms. Show that q becomes purely imaginary for $k_\parallel^2 > \varepsilon_\mathrm{f}\,\omega^2/c^2$.

d) Establish the conditions of continuity for the tangential components of $\vec{H}$ $(= \vec{B}$ here) and calculate the effective transmission amplitude $t = E_\mathrm{t}/E_\mathrm{i}$ and the effective reflection amplitude $r = E_\mathrm{r}/E_\mathrm{i}$. Discuss the maxima of the transmission coefficient $T = |t|^2$ in the case of normal incidence. For total reflection, $k_\parallel^2 > \varepsilon_\mathrm{f}\,\omega^2/c^2$, interpret the asymptotic behavior of T for thick films.

Problem 5.72 *Reflection of an electromagnetic wave at a conducting mirror*

A plane polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a nonpermeable medium of conductivity $\sigma \geq 0$ and a constant background susceptibility $\chi_m > 0$.

a) First consider the medium. Show that for harmonically time-varying fields, $\vec{E}(t) = \mathrm{Re}\,\vec{E}_\omega \mathrm{e}^{-\mathrm{i}\omega t}$ etc., the polarization $\vec{P} = \chi_m \vec{E}$ and the current density $\vec{j} = \sigma\vec{E}$ in Ampère's equation can be eliminated in favor of a complex dielectric permittivity,

$$\vec{\nabla} \times \vec{H}_\omega = \frac{-\mathrm{i}\omega}{c}\varepsilon(\omega)\vec{E}_\omega \qquad \text{with} \qquad \varepsilon(\omega) = \varepsilon_m + \frac{4\pi\mathrm{i}\sigma}{\omega}\,, \quad \varepsilon_m = 1 + 4\pi\chi_m\,.$$

b) The incident wave is partially reflected and absorbed by the medium. Choosing the z-axis perpendicularly to the flat surface, a suitable Ansatz for the electric field is given by

$$E_\omega(z) = E_\mathrm{i} \begin{cases} \mathrm{e}^{\mathrm{i}kz} + r\,\mathrm{e}^{-\mathrm{i}kz} & \text{for } z < 0 \text{ (empty space)}, \\ t\,\mathrm{e}^{\mathrm{i}qz}\,\mathrm{e}^{-\kappa z} & \text{for } z > 0 \text{ (medium)}. \end{cases}$$

Determine the wave numbers q, k as well as the decay rate κ by solving the corresponding wave equations.

c) Formulate appropriate matching conditions for the electromagnetic fields at the interface ($z = 0$) and determine the *reflection amplitude* r and the *transmission amplitude* t. Calculate the reflection coefficient $R = |r|^2$ and the transmission coefficient $T = 1 - R$.

d) Evaluate the time averaged Poynting vector

$$\langle S \rangle = \frac{1}{2}\,\mathrm{Re}\left(\frac{c}{4\pi}\vec{E}_\omega \times \vec{H}_\omega^*\right)\,.$$

in both half spaces and interpret your result.

e) Specialize your results for the case of good conductor $\sigma \gg \omega\varepsilon_m$, i.e., ε_m can be neglected, and discuss the decay rate κ and the reflection coefficient R. Argue that the displacement current is small compared the current density $\vec{j}$ in this case and show that the electromagnetic fields in the medium fulfill diffusion equations rather than wave equations.

f) In the opposite limit of a poor conductor, $\sigma \ll \omega\varepsilon_m$, the decay rate becomes large to the wavelength of the incident wave. Determine the absorption length κ^{-1} and the reflection coefficient R in this case.

Hints: The parts a–d can be solved independently of each other. The calculation of b) may be done for general complex $\varepsilon(\omega)$; the results of c) should be expressed in terms of q, k and κ, the Poynting vector in d) in terms of R and κ.

Problem 5.73 *Radiation loss of a harmonically oscillating charge*

a) A positive charge is attached to a spring (Fig. 4) in such a way that a radiating harmonic oscillator results. Show that small radiation losses with a minor reaction on a motion of the oscillator may be described by introducing a frictional force proportional to the third derivative of the elongation.

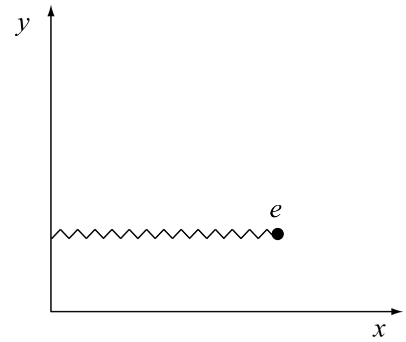


Fig. 4: A charge attached to a spring

b) Consider an isolated system which emits dipole radiation mainly with the frequency ω_0. Due to the radiation, the energy of the system is diminished permanently. This implies that the frequencies $\omega = \omega_0 + \Delta\omega$ adjacent to ω_0 are emitted by the system. $\Delta\omega$ is called the natural width of the emission line. Show that for a radiating harmonic oscillator of mass m and charge e, in case of weak damping, the natural line width is given by $\Delta\omega = \dfrac{2}{3}\dfrac{e^2\omega_0^2}{mc^3}$.

Problem 5.74 *Liénard-Wiechert potentials of a point charge moving with constant velocity*

For a point charge moving along an arbitrary trajectory with a constant velocity,

a) determine the Liénard-Wiechert potentials, and

b) derive the field intensities from the potentials.

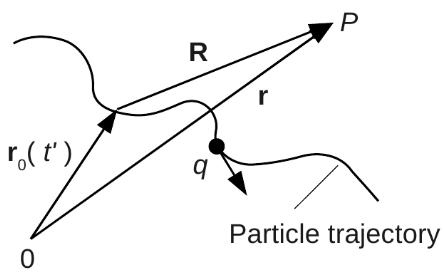


Fig. 5: A general particle trajectory

Problem 5.75 *Intensity of radiation*

Point charges $+e$ and $-e$ are moving along concentric circles with the radii a and $2a$, respectively, both with the same constant linear velocity v_0. Is the intensity of radiation (averaged over a cycle) of this dipole greater if both the charges are moving clockwise compared to the case when one charge is moving clockwise and the other counterclockwise?

Problem 5.76 *Linear antenna*

A current $J(r,t)$ is generated in a thin linear antenna of length d .
If the current is given in the form

$$J(r,t) = J_0 \sin\left(\frac{kd}{2} - k \mid z \mid\right) \delta(x)\delta(y)\vec{e}_z \mathrm{e}^{-\mathrm{i}\omega t}, \tag{5}$$

calculate the intensity of the radiation $\mathrm{d}I$ emitted into the solid angle $\mathrm{d}\Omega$, averaged over a period $T = \frac{2\pi}{\omega}$. Here, $\vec{e}_z$ is the unit vector in the direction of the z-axis.
Consider also the case when the wavelength is comparable to d.

Problem 5.77 *Drude-Hall effect*

As an extension of Drude's theory of conductors consider the induced current density $\vec{j}^{(\mathrm{ind})}(\vec{r},t)$ in the presence of a constant and uniform external magnetic field $\vec{B} = B\hat{e}_z$. Motivate the constitutive equation

$$\partial_t \vec{j}^{(\mathrm{ind})}(\vec{r},t) + \frac{1}{\tau}\vec{j}^{(\mathrm{ind})}(\vec{r},t) - \frac{e}{m^* c}\vec{B} \times \vec{j}^{(\mathrm{ind})}(\vec{r},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{r},t)\,, \qquad \omega_p^2 = \frac{4\pi n e^2}{m^*}\,,$$

where m^* denotes the effective mass of the conduction electrons (charge $-e$), τ a characteristic relaxation time, and n the density of conduction electrons. The characteristic frequency ω_p is referred to as plasma frequency.

a) Perform a temporal Fourier transform, convention $\vec{E}(\vec{r},\omega) = \int \mathrm{e}^{\mathrm{i}\omega t}\vec{E}(\vec{r},t)\mathrm{d}t$. Show that the response becomes local in the frequency domain,

$$j_k^{(\mathrm{ind})}(\vec{r},\omega) = \sigma_{kl}(\omega) \cdot E_l(\vec{r},\omega),$$

and determine the dynamic magneto-conductivity tensor $\sigma_{kl}(\omega)$.

b) Show that the dissipated power density $\dot{w} = \overline{\vec{j}(t) \cdot \vec{E}(t)}$ for harmonic driving $\vec{E}(t) = \mathrm{Re}[\vec{E}(\omega)\mathrm{e}^{-i\omega t}]$ averaged over many cycles $T = 2\pi/\omega$ may be written as

$$\dot{w} = \frac{1}{2}\mathrm{Re}[\vec{j}(\omega) \cdot \vec{E}(\omega)^*] = \frac{1}{4}E_i(\omega)^*[\sigma_{ik}(\omega) + \sigma_{ki}(\omega)^*]E_k(\omega)$$

Conclude that Symmetry, block structure, positive definite

c) Specialize to d.c. fields, i.e. $\omega = 0$, and discuss the Hall resistivity.

Problem 5.78 *Polaritons*

Consider the constitutive equation of the Lorentz-Drude model,

$$\partial_t^2 \vec{P}(\vec{x},t) + \frac{1}{\tau}\partial_t \vec{P}(\vec{x},t) + \omega_0^2 \vec{P}(\vec{x},t) = \frac{\omega_p^2}{4\pi}\vec{E}(\vec{x},t)\,,$$

with the relaxation time τ, characteristic frequency ω_0 and the plasma frequency ω_p.

a) Perform a spatio-temporal Fourier transform and determine the complex susceptibility $\chi(\omega)$, with $\vec{P}(\vec{k},\omega) = \chi(\omega)\vec{E}(\vec{k},\omega)$, as well as the dielectric function $\varepsilon(\omega) = 1 + 4\pi\chi(\omega)$.

b) Argue that the longitudinal modes follow from the zero of the dielectric function, $\varepsilon(\omega_*) = 0$, and determine the complex frequency ω_* in the case of weak damping.

c) Ignoring the damping, $\tau \to \infty$, determine the dispersion relation of the transverse modes.

d) Explain without calculation, in what frequency regime the damping is most important.

Problem 5.79 *Surface-plasmon polaritons*

Consider an interface between a metal characterized by a dielectric function $\varepsilon_1(\omega) = 1 - \omega_p^2/\omega^2$ and an ideal dielectric, $\varepsilon_2(\omega) = \varepsilon = const., \varepsilon > 1$. In each material the constitutive equations $\vec{D}_i = \varepsilon_i \vec{E}_i, \vec{B}_i = \vec{H}_i, i = 1, 2$ apply. The interface supports electromagnetic modes propagating along the interface (surface-plasmon polaritons). Taking the interface as the $z = 0$ plane and choosing the propagation of the mode as the x-direction, choose as an ansatz for the fields

$$\vec{E}_i(\vec{r}, t) = \vec{\mathcal{E}}_i \mathrm{e}^{\mathrm{i}(qx-\omega t)} \mathrm{e}^{-\kappa_i |z|}\,,, \qquad \vec{B}_i(\vec{r}, t) = \vec{\mathcal{B}}_i \mathrm{e}^{\mathrm{i}(qx-\omega t)} \mathrm{e}^{-\kappa_i |z|} \qquad \text{for } (i = 1, 2)\,,$$

with positive decay constants $\kappa_i > 0$.

a) Formulate appropriate continuity conditions for the amplitudes $\vec{\mathcal{E}}_i, \vec{\mathcal{B}}_i$ across the interface, $z = 0$.

b) Show that the magnetic fields are perpendicular to the interface and to the propagating direction, i.e. $\vec{\mathcal{B}} = (0, \mathcal{B}_y, 0)$.

c) Sketch the dispersion $\omega = \omega(q)$. Show that for short wavelengths $k \gg \omega_p/c$, the surface-plasmon polariton frequency approaches a constant ω_s whereas for long wavelength the dispersion is linear $\omega = c_s q$ to first order in q. Discuss the attenuation length $l_i = 1/\kappa_i$ in the metal and the dielectric as a function of frequency.

Problem 5.80 *Spectral line of radiation*

A plane electromagnetic wave propagates in space in the direction of the unit vector $\vec{n}$, and its electric field has the form

$$\vec{E}(\vec{r}, t) = \vec{E}_0\, e^{-|t'|/\tau} \cos\omega_0 t'\,,$$

where $t' = t - \vec{n} \cdot \vec{r}/c$, $\vec{E}_0$ is a constant vector, and the parameters ω_0 and τ satisfy the inequality $\omega_0 \tau \gg 1$. Find the spectral line $\varepsilon(\omega)$ of the radiation which propagates in the form of the given electromagnetic impulse. Find the width of the spectral line.

Problem 5.81 *Wave polarization*

Consider two monochromatically polarized electromagnetic waves

$$\vec{E}_1(\vec{r}, t) = \vec{E}_{01} \cos(\omega_1 t - \vec{k}_1 \cdot \vec{r})\,,$$
$$\vec{E}_2(\vec{r}, t) = \vec{E}_{02} \cos(\omega_2 t - \vec{k}_2 \cdot \vec{r})\,,$$

such that their amplitudes are equal, their wave vectors are parallel, their polarization vectors are perpendicular, and their frequencies satisfy the relation $|\omega_1 - \omega_2| \gg \omega_1 + \omega_2$. Determine the wave polarization as a result of the superposition of the given waves.

Relativity

Problem 6.82 *Transformations of the electromagnetic fields*

a) Show that the equations ("Electrodynamics before Faraday and Maxwell")

$$\begin{aligned} \vec{\nabla}_{\vec{x}} \cdot \vec{E} &= 4\pi\rho(\vec{x}, t) \\ \vec{\nabla}_{\vec{x}} \times \vec{E} &= \vec{0} \\ \vec{\nabla}_{\vec{x}} \cdot \vec{B} &= 0 \\ \vec{\nabla}_{\vec{x}} \times \vec{B} &= \frac{1}{c}\frac{\partial}{\partial t}\vec{E}(\vec{x}, t) + \frac{4\pi}{c}\vec{j}(\vec{x}, t) \end{aligned}$$

under the Galilean transformation

$$\vec{x}' = \vec{x} + \vec{v}t \qquad \text{and} \qquad t' = t$$

and with the following transformation of the fields

$$\vec{E}'(\vec{x}', t') = \vec{E}(\vec{x}, t) \qquad \text{and} \qquad \vec{B}'(\vec{x}', t') = \vec{B}(\vec{x}, t) + \frac{1}{c}\vec{v} \times \vec{E}(\vec{x}, t)$$

preserve their form when $\vec{j}$ and ρ are suitably (how?) transformed. What is the difference between these and Maxwell's equations? You may directly use the relation that $\dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t} - \vec{v} \cdot \vec{\nabla}$.

b) The relativistic transformation properties of the electromagnetic fields can be obtained through the Lorentz transformation of the Faraday tensor $F = F_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$. In any inertial system I the Faraday tensor F has the following representation:

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & \varepsilon\vec{B} \end{pmatrix}, \quad \text{with} \quad (\varepsilon\vec{B})_{ij} = \varepsilon_{ijk}B^k.$$

Calculate the components of the tensor for the system I' that is moving with velocity $\vec{v}$ relative to I. Attention: Using which Λ-matrix should $(F_{\alpha\beta})$ be transformed? Determine the transformed fields $\vec{E}'(\vec{E}, \vec{B})$ and $\vec{B}'(\vec{E}, \vec{B})$ through the comparison of $(F_{\alpha'\beta'})$ with $(F_{\alpha\beta})$.

Problem 6.83 *Relativistic Doppler effect*

a) Using the law of transformation of a wave 4-vector, determine the change in frequency (*Doppler effect*) and the change in the direction of the speed of light (*light aberration*) when going from one inertial system to another.

b) Analyze the obtained formula in the case $v \ll c$, where v is the absolute value of the relative velocity of the two inertial systems.

Problem 6.84 *Relativistic transformation of relative velocities*

a) In an inertial system K, two particles are moving with known velocities v_1 and v_2 along the same line but in opposite directions. Find an inertial system K' such that the velocities of the two particles are orthogonal in K'.

b) In the system K', what are the angles that the velocity vectors of the two particles make with the velocity vector of K?

Problem 6.85 *Momentum 4-vector*

In kinematics of reactions involving two particles

$$a + b \to c + d$$

it is common to use invariant variables s, t, and u defined as:

$$s = (p_a + p_b)^2, t = (p_a - p_c)^2, u = (p_a - p_d)^2$$

where p is the momentum 4-vector of the particle. Use s, t and u to express:

a) the energies E_c and E_d of the product particles in the laboratory system, and

b) the angle θ_{ac} between the momenta p_a and p_c in the laboratory system and in the center of mass system.

Problem 6.86 *Lorentz invariants*

Notation (both for this problem and for the next one ("Energy Momentum Tensor")): $\vec{E} = (E_x, E_y, E_z) = (E^1, E^2, E^3) = (-E_1, -E_2, -E_3)$, $\vec{B} = (B_x, B_y, B_z) = (B^1, B^2, B^3) = (-B_1, -B_2, -B_3)$.
The signature of the metric is (+,-,-,-).
The Levi-Civita Symbol $\varepsilon^{\alpha\beta\gamma\delta}$ has the following properties:

- $\varepsilon^{0123} = 1$
- Interchanging two indices changes the sign: $\varepsilon^{\ldots u \ldots v \ldots} = -\varepsilon^{\ldots v \ldots u \ldots}$

ε_{ijkl} can be calculated by $g_{i\alpha} g_{j\beta} g_{k\gamma} g_{l\delta} \varepsilon^{\alpha\beta\gamma\delta} = -\varepsilon^{ijkl}$.
We also use ε^{xyz} which corresponds to ε^{123}.
The Faraday tensor $\mathbf{F}$ is given by:

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\vec{E}^{\mathrm{T}]} \\ \vec{E} & (-{}^*\vec{B}) \end{pmatrix} \qquad i,j,k = x,y,z$$

in which ${}^*\vec{B}(= \varepsilon\vec{B})$ is the tensor $\varepsilon_{\alpha\beta\gamma} B^\gamma$.
The Maxwell-tensor ${}^*\mathbf{F}$ is given by:

$$({}^*F^{\alpha\beta}) = \frac{1}{2!}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{B}^{\mathrm{T}} \\ -\vec{B} & ({}^*\vec{E}) \end{pmatrix} \qquad i,j,k = x,y,z$$

The double contraction of the Faraday-tensor $\mathbf{F}$ with itself and with the Maxwell-tensor ${}^*\mathbf{F}$ yields two invariants of the electromagnetic field.

a) Calculate $\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}$ and $\frac{1}{4} F^*_{\alpha\beta} F^{\alpha\beta}$. Use the component representation of $(F^{\alpha\beta})$ and $({}^*F^{\alpha\beta})$.

b) Which relations between $\vec{E}$ and $\vec{B}$ are therefore independent of the reference system? Is it possible to transform a pure $\vec{E}$-field ($\vec{B} = \vec{0}$) into a pure $\vec{B}$-field only by a change of the reference system?

Problem 6.87 *Energy Momentum Tensor*

a) Give the components of the electromagnetic energy-momentum-tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\alpha}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

in terms of $\vec{E}$ and $\vec{B}$.

b) Calculate the divergence of T,

$$(\mathrm{div}T)^{\mu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$$

and show that

$$k^{\mu} := \frac{1}{c}F^{\mu}{}_{\nu}\,J^{\nu} = -(\mathrm{div}T)^{\mu}\;.$$

Problem 6.88 *Transformation of electromagnetic fields*

Apply the E-B-transformation to calculate the electromagnetic field of a moving charge q. To do this, set q at the origin of the system I, and transform its electromagnetic field from I to I'. Perform the calculation in the cgs system. Note: in the SI system $\varepsilon\vec{B}$ is replaced by $\varepsilon(c\vec{B})$.
What sign has to be used for the relative speed, so that q in I' moves with $\vec{v}$? Select the zero point in I' in such a way that the charge at $t' = 0$ is located at $\vec{x'} = 0$. Note: you must also transform the coordinates in order to correctly determine $\vec{E}'(x', \vec{t'} = 0)$ and $\vec{B}'(x', \vec{t'} = 0)$ from $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$! Illustrate the result.

Problem 6.89 *Relativistic Lorentz transformation*

An infinitesimal Lorentz transformation can be expressed as

$$x'^{\alpha} = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_{\beta}\;\;,$$

and its inverse transformation as

$$x^{\alpha} = \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)x'_{\beta}\;\;,$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon'^{\alpha\beta}$ are infinitesimal values. Show that $\varepsilon'^{\alpha\beta} = -\varepsilon^{\alpha\beta}$ and $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$.

Problem 6.90 *Lagrangian formalism for electrodynamics*

a) Develop the Lagrangian and the Hamiltonian equations for a classical non-relativistically moving charged particle in
   (i) a Coulomb field, and
   (ii) an external magnetic field.

b) For relativistically moving particles, covariance under a Lorentz transformation dictates that the Lagrangian function and the action integral have to be Lorentz scalars. Therefore, for the action integral we write

$$W = \int_{s_1}^{s_2} L(x_{\mu}, u_{\mu}, s)\mathrm{d}s$$

where L is a Lorentz scalar, $\{x_{\mu}\}$ is the space-time vector, $\{u_{\mu}\} = \{\mathrm{d}x_{\mu}/\mathrm{d}s\}$ is the four-velocity, and $\mathrm{d}s$ is chosen to be the arc length in Minkowski space. This L is the relativistically covariant Lagrangian. Alternatively, we can call the quantity cL/γ as the Lagrangian, in order to be consistent with our earlier definition of the Lagrangian, as $W = \int_{s_1}^{s_2} L(x_{\mu}, u_{\mu}, s)\mathrm{d}s = \int_{t_1}^{t_2} \frac{1}{\gamma}cL(x_{\mu}, u_{\mu}, s)\mathrm{d}t$.

As usual, for $v \ll c$, W has to transform into the action integral of the non-relativistic limit, i.e., $L(x_\mu, u_\mu, s)\mathrm{d}s \Longrightarrow L_{nonrel}(\vec{r}, \vec{v}, t)\mathrm{d}t$. With these concepts in mind, derive the relativistic Lagrange equations.

c) Set up the Lagrangian and the Lagrange equations for a relativistically-moving charged particle in an external electromagnetic field.

d) With the help of part (c) of this question, describe the motion of a relativistically-moving charged particle in
   (i) a uniform steady electric field $\vec{E}$ pointing in the direction of the x-axis, and
   (ii) parallel and uniform electric and magnetic fields pointing in the z-direction (Fig. **??**).

Problem 6.91 *Galilean and Lorentz transformations*

a) Prove the relations $\dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t} - \vec{v} \cdot \vec{\nabla}$ and $\vec{\nabla}' = \vec{\nabla}$ for the Galilean transformation used in the Problem set 3, question 1(a).

b) For the relativistic transformation used in the Problem set 3, question 1(b), find the parallel projection operator $\mathbf{P}_{\|} = \beta\beta^T/\beta^2$ and perpendicular projection operator $\mathbf{P}_{\perp} = (1 - \mathbf{P}_{\|})$ and use them to find the parallel and perpendicular components $\vec{E}_{\|}, \vec{E}_{\perp}, \vec{B}_{\|}, \vec{B}_{\perp}$ of the electric and magnetic field, respectively. Identify the $\mathbf{P}_{\|}$ in the Lorentz transformation matrix Λ.

Problem 6.92 *Relativistic length contraction*

Two sticks of rest-length L_0 are moving towards each other in the system S, such that the velocities of the sticks are opposite and have the same absolute value, and both the sticks are aligned along the direction of their velocities.
What is the length L of one of the sticks, as seen from the system S' connected to the other stick?

Problem 6.93 *Relativistic addition of velocities*

Consider three inertial systems K_i $(i = 1, 2, 3)$. Let v_{ij} be the velocity of the system K_i relative to an observer at rest in K_j. All velocities v_{ji} are parallel.

a) Show the relation

$$\beta_{12} + \beta_{23} + \beta_{31} = -\beta_{12}\beta_{23}\beta_{31}$$

where $\beta_{ij} = v_{ij}/c$.

b) What does the above relation reduce to in the nonrelativistic limiting case?